Quantum Error Correction - Theory and Hands-on

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9:00 - 12:00, 3. December 2024

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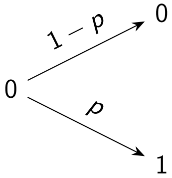
Outline

Classical noisy channel

- Send k-bit message across a noisy channel.
- Channel flips one bit independently with low probability p



- How do you protect from the noise?

Repetition code

- Repeat information, so e.g., send $x = 000$ instead of $x = 0$

$$x \xrightarrow{\text{noise}} \tilde{x}$$

- one receives:
  - 000 with probability $(1-p)^3$,
  - 100, 010, 001 each with probability $(1-p)^2 p$,
  - 011, 101, 110 each with probability $(1-p)p^2$, and
  - 111 with probability p^3.
- Let's say we receive $\tilde{x} = 010$
- Assuming at most one error occurred, we can take a majority vote to decode $x = 000$

Error correction process

$$x \xrightarrow{\text{encoding}} y \xrightarrow{\text{noise}} \tilde{y} \xrightarrow{\text{decoder}} \tilde{x}$$

- $\mathcal{E}(x) = y$ encodes a k bit message x, into an n bits
- A **codeword** is an element of the image of $\mathcal{E}$:
  Set of all codewords $C = \text{Im}(\mathcal{E})$
- E.g., 3-repetition code 0, $k = 1, n = 3$, we have $C = \{000, 111\}$
- If one or two bits are flipped, the error is **detectable**
- If all bits are flipped, the error is **undetectable** = **logical error**

Simple Parity-Check Code

Encoding: Given a 3-bit message (a, b, c), the parity-check code encodes it as:

$$E(a, b, c) = (a, b, c, z) \quad \text{where} \quad z = (a + b + c) \bmod 2$$

Properties:

- z indicates whether the sum of a, b, c is even $(z = 0)$ or odd $(z = 1)$.
- Any single-bit error can be detected:
  - If a, b, or c is flipped, $z \neq (a + b + c) \bmod 2$.
  - If z is flipped, it no longer corresponds to the parity of a, b, c.

Limitation: Errors cannot be corrected.

Hamming Codes - The Idea

- Hamming introduced a method to **correct errors** using parity-check bits.
- Example: A 4-bit message $x = (a, b, c, d)$ with 3 parity-check bits:

$$z_1 = a + b + d, \quad z_2 = a + c + d, \quad z_3 = b + c + d \pmod 2$$

- Encoded message:

$$y = E(a, b, c, d) = (a, b, c, d, z_1, z_2, z_3)$$

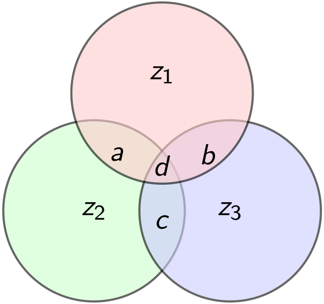
Parity-check Visualization

Error Detection:

- If a single bit is flipped, certain parity-checks will fail.
- Example:
  - If a is flipped, z_1 and z_2 will fail, while z_3 remains valid.

Error Correction:

- The pattern of failed parity-checks indicates the position of the flipped bit.
- Example: Flip in d causes all z_1, z_2, z_3 to fail.



Introduction to Linear Codes

- Linear codes generalize the concept of transmitting a message with parity-check bits.
- A **linear code** uses a matrix $\boldsymbol{G}$—called the **generator matrix**—to encode the message:

$$\boldsymbol{y} = \boldsymbol{G}\boldsymbol{x}.$$

- The message $\boldsymbol{x}$ has length k, and is supplemented with m parity-check bits such that the encoded message $\boldsymbol{y}$ has length $n = k + m$.
- The generator matrix $\boldsymbol{G}$ can be written as:

$$\boldsymbol{G} = \begin{pmatrix} I_k \\ \hline \boldsymbol{A} \end{pmatrix}$$

  - I_k: $k \times k$ identity matrix (reproduces the message bits),
  - $\boldsymbol{A}$: $m \times k$ matrix (defines parity-check operations).

Example: Generator Matrix of the Hamming Code

For the **[7,4]-Hamming code**, the generator matrix is:

$$\boldsymbol{G} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

Encoding a message $\boldsymbol{x} = (a, b, c, d)^T$:

$$\boldsymbol{Gx} = \begin{pmatrix} a \\ b \\ c \\ d \\ a+b+d \\ a+c+d \\ b+c+d \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \\ z_1 \\ z_2 \\ z_3 \end{pmatrix}.$$

Properties of Generator Matrices

Code is defined as image of G:

1. The codewords are the set of all linear combinations of the columns of $\boldsymbol{G}$.
2. To find all the codewords, just calculate all the $\boldsymbol{y}$ of the form $\boldsymbol{y} = a_1\boldsymbol{g_1} + \cdots + a_k\boldsymbol{g_k}$, where $\boldsymbol{g_i}$ is the i^{th} column of G and $a_1, \ldots, a_k \in \{0, 1\}$.
3. Elementary row and column operations on $\boldsymbol{G}$ do not change the code.
4. Using Gaussian elimination, $\boldsymbol{G}$ can always be transformed into the standard form:

$$\boldsymbol{G} = \begin{pmatrix} I_k \\ \hline \boldsymbol{A} \end{pmatrix}.$$

Parity-Check Matrix

An equivalent representation of a linear code is the **parity-check matrix $\boldsymbol{H}$**:

$$\boldsymbol{H}\boldsymbol{y} = \mathbf{0},$$

- $\boldsymbol{H}$ is an $m \times n$ matrix,
- $\boldsymbol{y}$ is a codeword if and only if $\boldsymbol{H}\boldsymbol{y} = \mathbf{0}$.
- set of all codewords $C = \mathrm{Ker}(H)$

For the [7,4]-Hamming code:

$$\boldsymbol{H} = \left(\begin{array}{cccc|ccc} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{array} \right).$$

- A received word $\widetilde{\boldsymbol{y}}$ can be checked for errors by evaluating $\boldsymbol{H}\widetilde{\boldsymbol{y}}$.
- Errors can often be located and corrected using this method.

Generator Matrix and Parity-Check Matrix

For a generator matrix of the form:

$$\boldsymbol{G} = \left(\frac{I_k}{\boldsymbol{A}}\right)$$

the corresponding parity-check matrix can be written as:

$$\boldsymbol{H} = \begin{pmatrix} \boldsymbol{A} & I_m \end{pmatrix}.$$

Why Use the Parity-Check Matrix?

- A received vector $\widetilde{\boldsymbol{y}} = \boldsymbol{y} + \boldsymbol{e}$ combines the original codeword $\boldsymbol{y}$ and an error vector $\boldsymbol{e}$.
- Applying the parity-check matrix $\boldsymbol{H}$ gives:

$$\boldsymbol{H}\widetilde{\boldsymbol{y}} = \boldsymbol{H}(\boldsymbol{y} + \boldsymbol{e}) = \boldsymbol{H}\boldsymbol{e}.$$

- The result $\boldsymbol{s} = \boldsymbol{H}\boldsymbol{e}$ is known as the **syndrome**.
- The syndrome identifies errors by revealing violated parity-check equations: $s_i = 1$ indicates a violation.

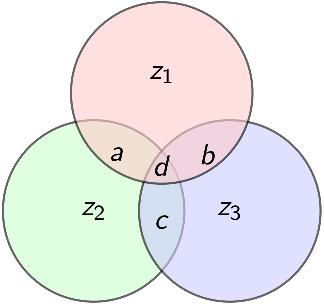
Decoding:

- Decoding finds the most probable error $\boldsymbol{e}$ that explains the syndrome $\boldsymbol{s}$.
- A violated parity-check equation points to specific bits involved in the error.

Syndrome Table for Error Correction

The following table shows the bit we choose to correct for each of the 8 possible syndromes

Syndrome	000	100	010	001	110	101	011	111
Correction	$\emptyset$	z_1	z_2	z_3	a	b	c	d



In quantum error correction, the syndrome can/has to be measured without disturbing the quantum state

Decoding linear codes

Decoding consists of finding the original message given its noisy encoded version.

- There are 2^n possible syndromes.
- We define an **efficient decoder** as an algorithm that accomplishes this task in polynomial time in n.

Given the parity-check matrix $\boldsymbol{H}$. Let's assume errors follow a certain distribution $P(\boldsymbol{e})$.

- Given the received syndrome s, we want to find the most likely error e.
- The **goal of an ideal decoder**:

  Find the vector $\boldsymbol{e}$ that **maximizes** the probability $P(\boldsymbol{e} \mid \boldsymbol{s})$.

Applying Bayes' Rule

Using Bayes' rule, we can write:

$$P(\boldsymbol{e} \mid \boldsymbol{s}) = \frac{P(\boldsymbol{s} \mid \boldsymbol{e})P(\boldsymbol{e})}{P(\boldsymbol{s})}.$$

- $P(\boldsymbol{s})$ does not depend explicitly on $\boldsymbol{e}$, and can be ignore for solving the maximization problem over $\boldsymbol{e}$
- Any valid error $\boldsymbol{e}$ must satisfy $\boldsymbol{He} = \boldsymbol{s}$, so:

$$P(\boldsymbol{s} \mid \boldsymbol{e}) = \begin{cases} 1, & \text{if } \boldsymbol{He} = \boldsymbol{s} \\ 0, & \text{otherwise.} \end{cases}$$

Our optimization problem then becomes:

$$\max_{\boldsymbol{e} \in \{0,1\}^n} P(\boldsymbol{e}) \quad \text{subject to} \quad \boldsymbol{He} = \boldsymbol{s}.$$

Special Case: Independent Errors

- In the case where errors are iid, we have:

$$P(\boldsymbol{e}) = \prod_{i=1}^{n} P(e_i).$$

- Let $P(e_i = 1) = p$ and $P(e_i = 0) = 1 - p$. Then:

$$P(\boldsymbol{e}) = p^{|\boldsymbol{e}|}(1-p)^{n-|\boldsymbol{e}|},$$

  where $|\boldsymbol{e}|$ is the Hamming weight of $\boldsymbol{e}$.
- For $p < 0.5$, the probability $P(\boldsymbol{e})$ increases when the weight of $\boldsymbol{e}$ decreases. Therefore, our optimization problem reduces to finding the error of minimum weight that satisfies $\boldsymbol{He} = \boldsymbol{s}$:

$$\min_{\boldsymbol{e} \in \{0,1\}^n} |\boldsymbol{e}| \quad \text{subject to} \quad \boldsymbol{He} = \boldsymbol{s}.$$

Maximum A Posteriori (MAP) Decoding and Its Challenges

MAP Decoder: Any decoder that explicitly solves

$$\max_{\boldsymbol{e}\in\{0,1\}^n} P(\boldsymbol{e} \mid \boldsymbol{s}),$$

is called MAP decoder, and is considered an *ideal decoder*.

Challenges:

- A naive approach requires searching all 2^n possible error vectors, leading to exponential time complexity.
- The MAP decoding problem is **NP-complete**, meaning no general polynomial-time algorithm is likely to exist.

Efficient Decoding for Special Codes:

- Certain structured codes (e.g., Hamming codes, repetition codes) allow polynomial-time decoding.

Heuristic Approaches to MAP Decoding

Heuristics: Approximate solutions for MAP decoding that are efficient and perform well in practice.

Belief Propagation Algorithm:

- An iterative, linear-time algorithm.
- Exploits the factorization of $P(\boldsymbol{e} \mid \boldsymbol{s})$ over a graph (e.g., Tanner graph).
- Widely used in classical error-correction and also in quantum error correction.

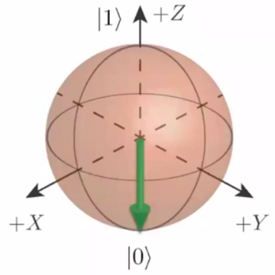
Outline

Quantum Logic

- Quantum bit/Qubit

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

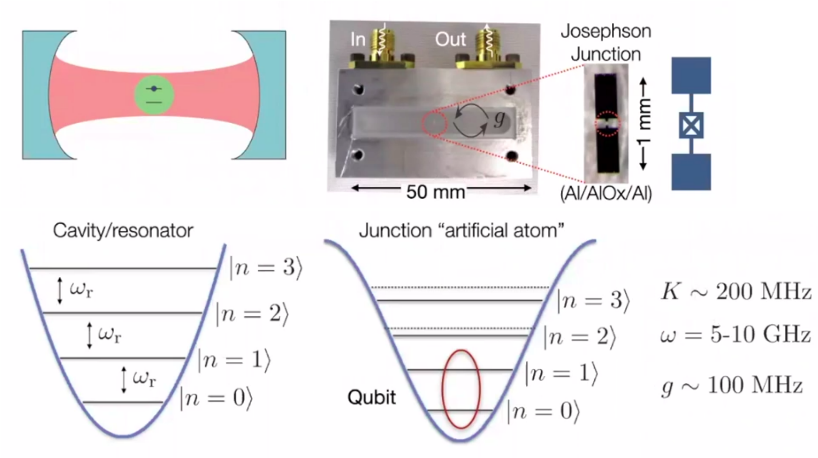
$$|\alpha|^2 + |\beta|^2 = 1$$



- Universal logical operations, gates, unitaries: Hadamard, S-gate, T-gate, CNOT

- **Measurements:**
  - Set of operators $\{M_i\}$ such that $\sum_i M_i^\dagger M_i = I$
  - Probability of outcome i is $p(i) = \langle\psi|M_i^\dagger M_i|\psi\rangle$
  - State after obtaining outcome i is $\frac{M_i|\psi\rangle}{\sqrt{p(i)}}$

Hardware: Superconducting Circuits

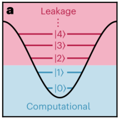


$$\hat{H} = \omega_r \hat{a}^\dagger \hat{a} \qquad \hat{H} \sim \omega_q \hat{b}^\dagger \hat{b} - K \hat{b}^{\dagger 2} \hat{b}^2$$

PRA 69, 062320 (2004)

Sources of Quantum Noise / Errors

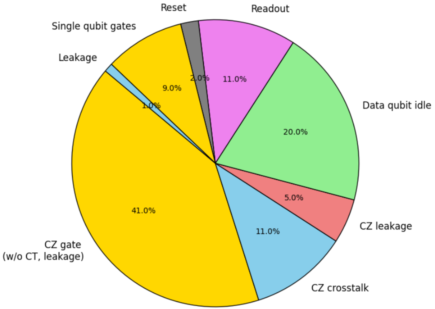
- **Decoherence:**
  - T1 relaxation: Energy decay from the $|1\rangle \to |0\rangle$
  - T2 dephasing: Loss of phase coherence in superposition states.
- **Gate Errors:** Imperfect implementation of quantum gate operations, leading to inaccuracies.
- **Measurement Errors:** Errors during the readout of qubit states, resulting in incorrect outputs.
- **Cross-Talk:** Interference between neighboring qubits during operations, reducing fidelity.
- **Leakage Errors:** Qubits transitioning to higher energy states outside the computational basis.

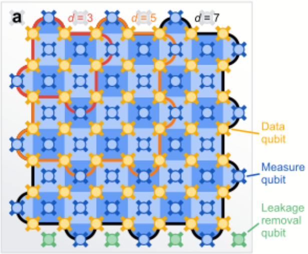


- **Stray Interactions:** Unintended couplings during gate operations.
- **Idle Errors:** Errors occurring while qubits remain idle due to environmental interactions.
- **External Noise:** Electromagnetic interference or cosmic rays.

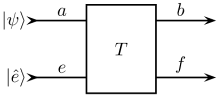
Error budget distribution

distance-5 surface code on 72-qubit processor (arXiv:2408.13687v1)





Noisy Quantum Channels



- A noisy quantum channel introduces errors during transmission or processing.
- Examples of noise effects:
  - Degradation of quantum states.
  - Reduction in entanglement and coherence.
  - Significant impact on fidelity and performance.
- Kraus operators provide a powerful framework to describe and analyze noise in quantum systems.

Kraus Operators

- Describe the evolution of quantum states in open systems.
- Evolution of a density matrix (ρ) under Kraus operators:

$$\rho' = \sum_i K_i \rho K_i^\dagger$$

- Completeness relation ensures trace preservation:

$$\sum_i K_i^\dagger K_i = I$$

- These operators model common errors such as bit-flip, phase-flip, and depolarization.

Bit-Flip Channel

- Models noise where qubits flip between $|0\rangle$ and $|1\rangle$ with probability p.
- Quantum state evolution:

$$\mathcal{T}(\rho) = (1 - p)\rho + pX\rho X^\dagger$$

- Kraus operators:

$$K_0 = \sqrt{1 - p}\, I, \quad K_1 = \sqrt{p}\, X$$

Depolarizing Channel

- Randomizes the qubit state with probability p.
- Channel action:

$$\mathcal{E}(\rho) = (1-p)\rho + \frac{p}{3}(X\rho X^\dagger + Y\rho Y^\dagger + Z\rho Z^\dagger)$$

- Kraus operators:

$$K_0 = \sqrt{1-p}\, I, \quad K_1 = \sqrt{\frac{p}{3}}\, X, \quad K_2 = \sqrt{\frac{p}{3}}\, Y, \quad K_3 = \sqrt{\frac{p}{3}}\, Z$$

Amplitude Damping Channel

- Models energy dissipation, such as photon loss.
- Channel action:

$$\mathcal{E}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger$$

- Kraus operators:

$$E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}$$

Quantum Error Correction

- Quantum error correction (QEC) is essential for protecting quantum information against noise and decoherence.
- The primary goals of QEC are:
  - Detect errors without disturbing the quantum information.
  - Correct errors to restore the original quantum state.
  - Ensure fault-tolerant quantum computation.

Errors in quantum computers

- Classically, bits can flip (or be erased).
  i.e., $0 \to 1$ and $1 \to 0$ with some probability p.
- Qubits have a larger state space, so more things can go wrong.
  - Any operation that can be considered a gate can also introduce an error.
  - Examples include Pauli errors (X, Z, Y).

$X\|0\rangle = \|1\rangle$ $X\|1\rangle = \|0\rangle$ **Bit flip**	$Z\|0\rangle = \|0\rangle$ $Z\|1\rangle = -\|1\rangle$ **Phase flip**	$Y\|0\rangle = i\|1\rangle = iXZ\|0\rangle$ $Y\|1\rangle = -i\|0\rangle = iXZ\|1\rangle$ **Bit & phase flip**

The Most Important Fact About QEC

- Errors are inherently continuous (analog). How can we hope to correct these?
- Suppose some error E introduces a relative phase:

$$E|\psi\rangle = \alpha|0\rangle + e^{i\delta}\beta|1\rangle$$

- The angle δ could (in principle) be infinitesimal.
- Any error can be written as discrete Pauli errors with continuous coefficients:
  - This is because the Pauli matrices (+ the Identity) span $\mathbb{C}^{2\times 2}$.
  - For any E and ψ:

$$E|\psi\rangle = (e_0 I + e_1 X + e_2 Y + e_3 Z)|\psi\rangle$$

- But the coefficients e_i could still be infinitesimal, in principle.

The Most Important Fact About QEC

- Measurement **turns** continuous errors into discrete errors.
- Suppose we measure the error state using operators $\{M_i\}$:

$$E|\psi\rangle = (e_0 I + e_1 X + e_2 Y + e_3 Z)|\psi\rangle$$

- Then, with probability $p(i)$, the state collapses to: $\frac{M_i E|\psi\rangle}{\sqrt{p(i)}}$
- This process collapses the superposition and reduces the continuous coefficients to a global phase, which is irrelevant.
  - For example, we could choose M_i such that: $E|\psi\rangle \to \eta_i \sigma_i |\psi\rangle$
  - Here, $\sigma_i \in \{I, X, Y, Z\}$ is a discrete error that can be corrected.
  - The coefficient $\eta_i \in \mathbb{C}$ is continuous but represents a global phase and hence does not matter.

Outline

Classical error correction: The repetition code

- A key concept in error correction is adding **redundancy**.
- For example, given a bit, we can make three copies of it:
  - $0 \to 000, \quad 1 \to 111$
  - This is known as the (classical) **repetition code**.
  - The idea is very simple: If an error occurs on one bit only, we can correct it by looking at the other two bits and taking a majority vote.
- Given the classical repetition code, we might try to do the same with qubits, i.e. map

$$|\psi\rangle \to |\psi\rangle|\psi\rangle|\psi\rangle$$

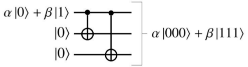
- This is not possible due to the **"no cloning theorem"**

QEC: Can We Add Any Redundancy?

- From the no-cloning theorem, we know it is **not** possible to make exact copies of a quantum state as in the classical repetition code.
- Can we copy information?
- **Claim:** We can "copy basis information" in the following sense:

$$\alpha|0\rangle + \beta|1\rangle \rightarrow \alpha|000\rangle + \beta|111\rangle$$

- Note that this encoding circuit entangles the "input" qubit with two other qubits.



- Errors in quantum computers are often caused by qubits entangling with their **environment**.

Repetition code for bit flip errors

- The encoding $\alpha|0\rangle + \beta|1\rangle \rightarrow \alpha|000\rangle + \beta|111\rangle$ gives us **redundancy**. Now what?
- We need to check which errors (if any) occured in the encoded state.
- We do this by (projective) measurements. What projections should we apply to find out what happened?
- There are four possible things that can happen:

No qubit was flipped.	$P_0 = |000\rangle\langle000| + |111\rangle\langle111|$
The first qubit was flipped.	$P_1 = |100\rangle\langle100| + |011\rangle\langle011|$
The second qubit was flipped.	$P_2 = |010\rangle\langle010| + |101\rangle\langle101|$
The third qubit was flipped.	$P_3 = |001\rangle\langle001| + |110\rangle\langle110|$

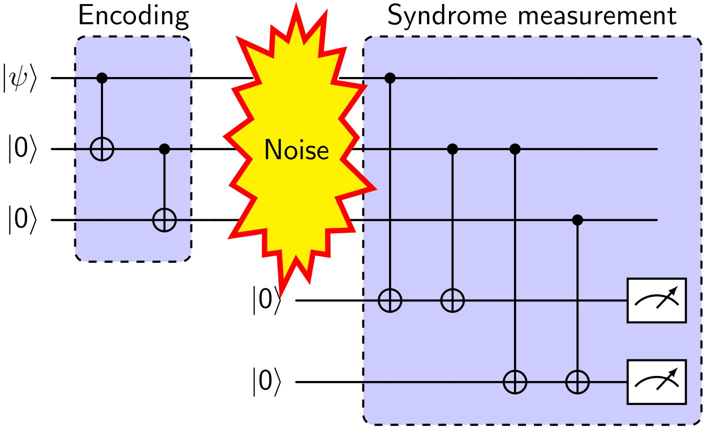
Turning the table

- By measuring these operators, we learn what errors (if any) occurred.
- Since we know which error occurred, we can **correct** it.

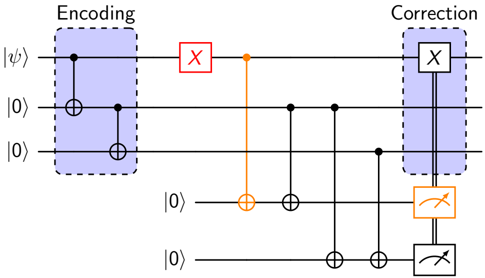
Syndrome measurement	Meaning	Correction operator
$P_0 = \|000\rangle\langle 000\| + \|111\rangle\langle 111\|$	No qubit was flipped.	I
$P_1 = \|100\rangle\langle 100\| + \|011\rangle\langle 011\|$	The first qubit was flipped.	X_0
$P_2 = \|010\rangle\langle 010\| + \|101\rangle\langle 101\|$	The second qubit was flipped.	X_1
$P_3 = \|001\rangle\langle 001\| + \|110\rangle\langle 110\|$	The third qubit was flipped.	X_2

- But since measurement **collapses** the state, we need to use ancilla qubits for syndrome measurement.

Bitflip Repetition Code: circuit implementation

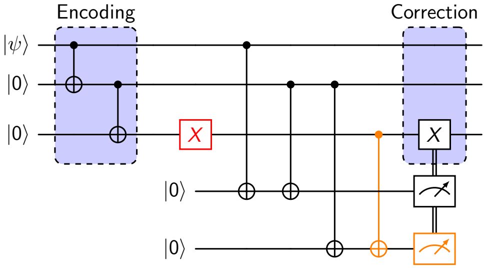


Bitflip Repetition Code: circuit implementation



Syndrome	Correction
00	I
10	X_1
01	X_3
11	X_2

Bitflip Repetition Code: circuit implementation



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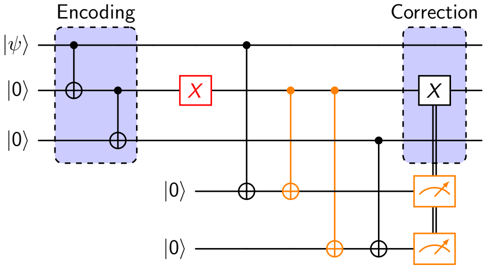
Bitflip Repetition Code: circuit implementation



Syndrome	Correction
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10	X_1
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Outline

Stabilizer Formalism: The Pauli Group

- Stabilizer codes are a class of quantum error correcting codes defined by commuting sets of Pauli operators, called the **stabilizer generators**
- Define the Pauli group
  $G_1 = \{\pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm, Z, \pm iZ\} = \langle X, Y, Z \rangle$
- It is enough to consider X, Z together with the prefactors $\pm i$ because $Y = iXZ$
- Any qubit unitary can be written as a linear combination of elements of G
- We also define G_n as n-fold tensor products of elements in G_1
- Notation: $Z_1 Z_3 \equiv Z \otimes I \otimes Z$

The Stabilizer Group - definition [1]

- Consider some subgroup $S \subset G_n$, where all elements **commute**
- Let V_S be a $2k$-dimensional subspace of n-qubit states defined by $s|\psi\rangle = +1|\psi\rangle\, \forall s \in S, \forall |\psi\rangle \in V_s$
- This defines a $[[n,k]]$ stabilizer code, which encodes k logical qubits into n physical qubits
- We say S is the **stabilizer group** of V_S, and conversely call V_S the **codespace** stabilized by S
- Example: 3-qubit repetition code [[3, 1]]:
  - Stabilizer group generators: $S = \langle Z_1 Z_2, Z_2 Z_3 \rangle$
  - Codespace: $V_S = \{\alpha|000\rangle + \beta|111\rangle \,|\alpha,\beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1\}$

[1] D. Gottesmann, Stabilizer codes and quantum error correction, arXiv:quant-ph/9705052

Stabilizer Codes: Error Detection and Correction

- Say some error $g \in G_n$ occurs on $|\psi\rangle \in V_S$. Since elements of G_n either **commute** or **anti-commute** with each other, g will either commute or anti-commute with each stabilizer in S
- If it **anti-commutes** with at least one stabilizer, it is a **detectable error**
- If it **commutes** with all stabilizers and is not itself a stabilizer, it is a **non-detectable** error (logical operator)
- Example: 3-qubit repetition code:
  - $\{X_1, Z_1 Z_2\} = 0$ so X_1 is a detectable error
  - $[X_1 X_2 X_3, Z_1 Z_2] = [X_1 X_2 X_3, Z_2 Z_3] = 0$ so $X_1 X_2 X_3$ is a non-detectable (i.e. logical error). In fact it is logical X in this code
- Measuring all the stabilizer generators on logical state $|\psi\rangle$ will give us a **syndrome** that we then use to apply the corresponding correction (analogous to classical parity checks) (see 3-qubit repetition code circuit from earlier)

Calderbank-Shor-Steane (CSS) codes

- In general, stabilizer generators can have mixed elements e.g. $X_1 Z_2 Z_3 X_4 \ldots$
- CSS codes are a "nice" type of stabilizer codes built by taking the parity check matrices H_X and H_Z of 2 classical codes C_1 and C_2 to define the X and Z stabilizers respectively. The generators are thus only pure X or pure Z operators (see hands-on session)
- Syndrome measurement: to measure a qubit in the X basis we need to apply a Hadamard transform to the qubit since $\langle\psi| HZH |\psi\rangle = \langle\psi| X |\psi\rangle$

Outline

Transversal gates in CSS codes

- Transversal gates, gates that can be written as a tensor product of gates inside each **code block**, are a type of fault-tolerant gates
- All Clifford gates are transversal in CSS codes
- For example in some CSS codes the *CNOT* gate on logical qubits 1 and 2 can be implemented by applying a *CNOT* gate between each homologous qubit of code blocks 1 and 2

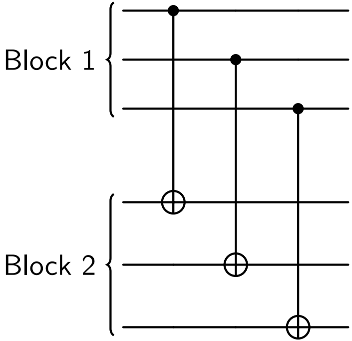


Figure: Transversal CNOT gate implementation for a 3-qubit CSS code

Transversal gates and error spread

An operation is said to be **fault-tolerant** if it does not increase the weight of an error $w(e)$ = the number of qubits that e affects within one **code block**.

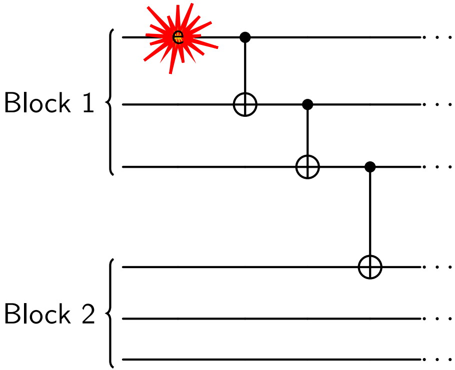


Figure: Non-fault tolerant CNOT gate implementation for a 3-qubit code

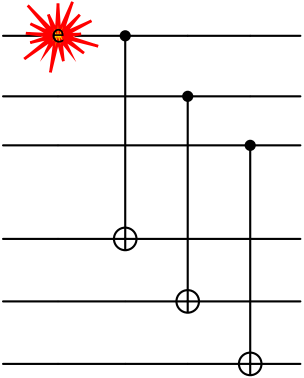


Figure: Transversal CNOT gate

Eastin-Knill Theorem

- **Theorem**: There is no non-trivial local-error-detecting quantum error correcting code that admits a universal set of transversal gates[2]. :(
- But transversal is not the only fault-tolerant construction!

[2] Eastin, B., Knill, E. (2009). Restrictions on transversal encoded quantum gate sets. Physical review letters, 102(11), 110502.

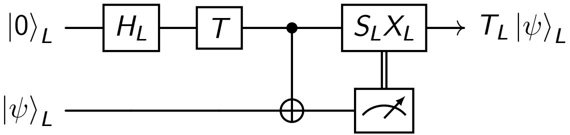
Universal Quantum Computing with Logical Qubits

Knill-Gottesman: Clifford-circuits efficiently simulable

- Generated by $\{H, S, \mathrm{CNOT}\}$ gates
- Many codes allow transversal implementation

Non-Clifford (e.g. T-gate), required for universal gate set.

- **Eastin-Knill**: No transversal implementation for CSS codes
- Requires magic state preparation and teleportation



Fault-tolerant T-gate

Goal: Apply logical T-gate to state $|\psi\rangle_L = a|0\rangle_L + b|1\rangle_L$
Need ancilla qubit. T gate is applied transversally $\rightarrow$ Does not correspond to logical T-state.

State before measurement

$$\frac{1}{\sqrt{2}}(a|0\rangle + be^{i\pi/4}|1\rangle)|0\rangle + (b|0\rangle + ae^{i\pi/4}|1\rangle)|1\rangle$$

If we measure $|0\rangle$, we are done, otherwise apply correction SX.
Preparation of ancilla has to be done fault-tolerantly!

Threshold Theorem

- Reliable quantum computation is possible if the physical error rate p is below a certain **threshold** p_{th}.
- For $p < p_{\text{th}}$, error is **exponentially suppressed** as we scale the code.

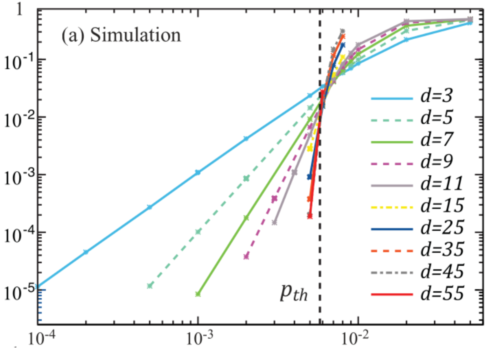


Figure: Exponential suppression as we scale the code

Outline

Surface Code - Introduction

- 2D stabilizer code proposed by Kitaev et al. [7]
  Belongs to the class of CSS codes
- Pauli-Z and Pauli-X type checks
- Planar graph connectivity
  Ideal for superconducting circuits
- High threshold ($\sim$1%) against noise
- Parallel syndrome extraction

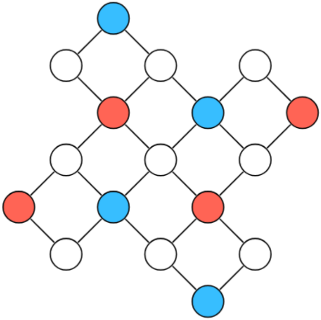


Figure: Surface code with 9 data and 8 ancilla qubits [15].

Surface Code - Stabilizers

Stabilizers at the interior of the surface check 4 qubits at a time. For each group (called *plaquette*) we have:

$$S_X^{(i)} = X_i X_{i+1} X_{i+2} X_{i+3} \qquad S_Z^{(i)} = Z_i Z_{i+1} Z_{i+2} Z_{i+3}$$

Detect an odd number of X/Z errors per plaquette.

Order of CNOT gates matters to avoid *hook errors*.

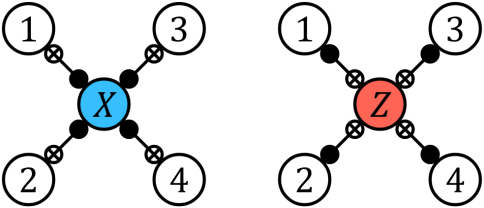


Figure: Pauli-X and Pauli-Z type stabilizers for the surface code. CNOT gate schedule measuring syndrome indicated by vertex index [15].

Surface Code - Error Classification

Only need to correct Pauli errors: $E = P_1 \otimes \ldots \otimes P_n$ where $P_i \in \{I, X, Y, Z\}$

Detectable Errors

- Anti-commute with stabilizers: $\exists S \in \mathcal{S} : SE = -ES$
- Example: Single-qubit errors

Undetectable errors

- Product of stabilizers: $E = S_1 \ldots S_n,\ S_i \in \mathcal{S}$
- Logical operators: Normalizers of $\mathcal{S}$

Beauty of surface code: Errors have *topological* interpretation!

Surface Code - Detectable Errors

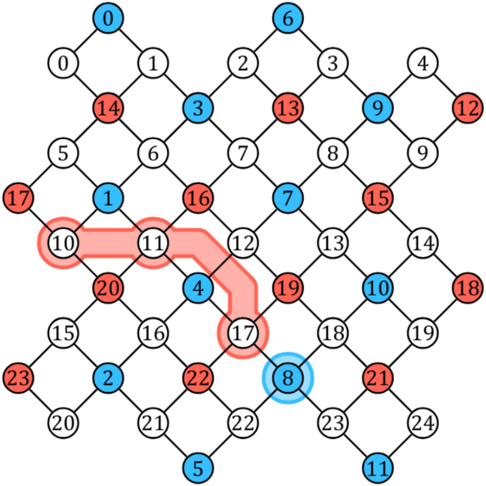


Figure: Detectable chain of Pauli-Z errors [15].

Surface Code - Detectable Errors

Error Chain Properties:

- Errors manifest as **chains** on surface
- Chain endpoints flagged by syndromes:
  - One syndrome if chain ends at **boundary**
  - Two syndromes for **interior** chains
- Pauli-Y triggers 4 syndromes Equivalent to X and Z errors

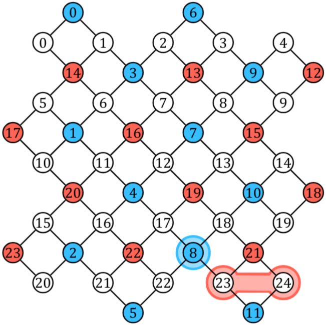


Figure: Detectable chain of Pauli-Z errors [15].

Surface Code - Undetectable Errors

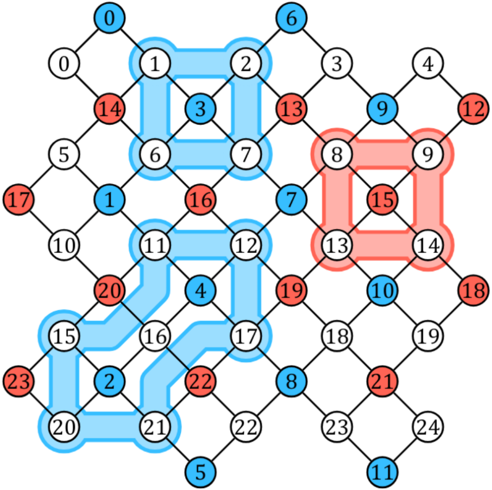


Figure: Undetectable errors which are products of stabilizers generators [15].

Surface Code - Logical Gates

Logical Gates Properties:

- Connect opposite borders
- Unique up to stabilizer product
- Anti-commuting logical operators cross odd number of times

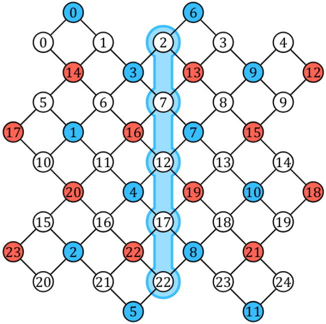


Figure: Chain of Pauli-X forming logical X_L operator [15].

Surface Code - Logical Gates

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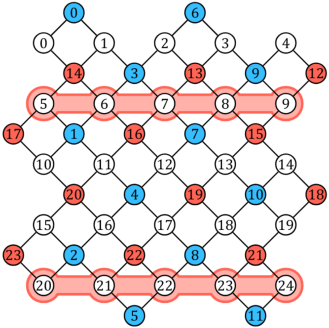

Figure: Chains of Pauli-Z forming logical Z_L operator [15].

Surface Code - Logical Gates

Logical Gates Properties:

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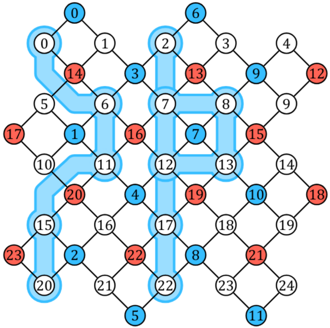


Figure: Equivalent logical X_L chains [15].

Surface Code - Code Distance

Question: How many errors can we correct?

For d^2 data qubits, shortest logical error chain has length d.
$\rightarrow$ We can correct up to $\left\lfloor \frac{d-1}{2} \right\rfloor$ errors.

The surface code is a $[[d^2, 1, d]]$ CSS code with code distance d.

Surface Code - Entangling Gates

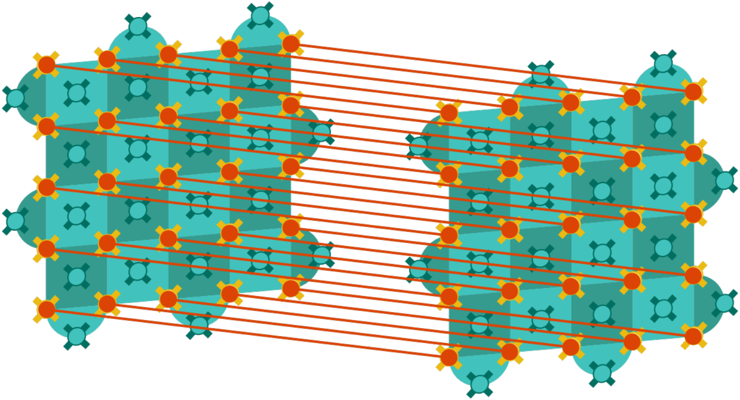


Figure: **Transversal** logical CNOT with pairwise matching physical qubits. Suitable for neutral atom or ion-trapped architectures where qubits can be moved [4].

Surface Code - Entangling Gates

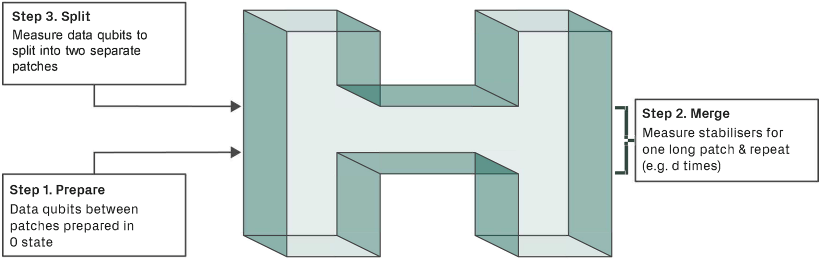
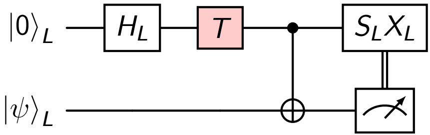


Figure: Logical CNOT through **lattice surgery**. Involves an ancilla surface code patch and stabilizer measurement along adjacent surface edges [4].

Surface Code - T-gate (via state-injection)

T-gate prepared via state teleportation.

Need **magic state**: $T_L \left|+\right\rangle_L$



Protocol for faulty state-injection [12]:

- Prepare physical qubit in state $\left|\psi\right\rangle$
- Initialize small distance surface code (e.g. $\hat{d} = 3$) in $\left|0\right\rangle_L$
- Spread state via CNOT operations
- Protect state with syndrome measurement rounds
- Grow to target distance: $\hat{d} \to d$

Not fault-tolerant: Low-distance $\hat{d}$ allows for errors

Surface Code - State injection example

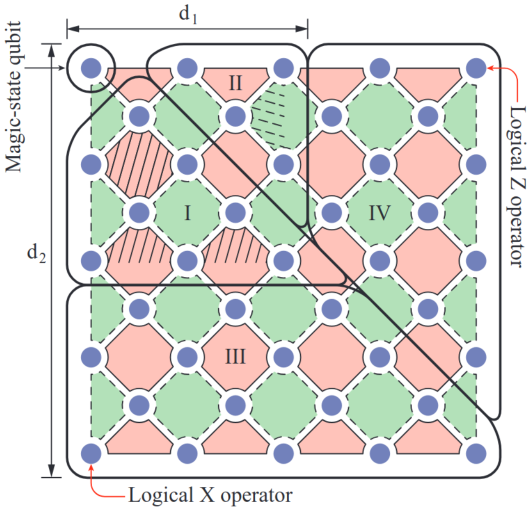


Figure: Preparation of magic state by preparing a single physical qubit and growing the surface code distance [12].

Surface Code - Magic State Distillation

Muller-Reed $[[15, 1, 3]]$**-code**: Smallest code with transversal T-gate

Example: 15-to-1 protocol

1. Encode by measuring stabilizers
2. Apply transversal (faulty) T-gate
   Surface code: Via state-injection
3. Measure Stabilizers
   Detect up to weight-3 errors
4. Discard and repeat, if errors detected

Output: Magic state $T_L \left|+\right\rangle_L$

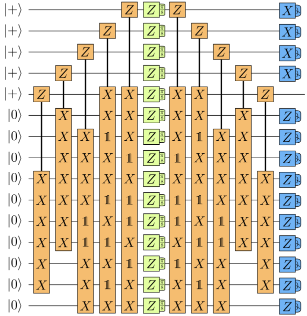


Figure: 15-to-1 magic state distillation protocol [13].

If error probability of T-gate is p_{in}, success probability is $p_{out} = 35p_{in}^3$.
Further distillation rounds can use teleportation for transversal T_L.

Fault-Tolerant Quantum Architecture

Based on surface code with Clifford+T gate set. Gates are implemented using fault-tolerant lattice surgery.

Core Processor Components

1. **Memory Fabric**
   - Data storage
   - Performs logical operations
2. **Magic State Buffer**
   - Stores prepared T-states
   - Enables on-demand T-gates
3. **Magic State Factory**
   - 15:1 distillation protocol
   - Continuous state preparation

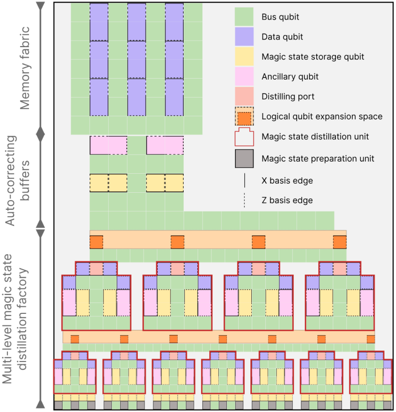


Figure: FTQC architecture [16].

Fault-Tolerant Quantum Computing - T-count

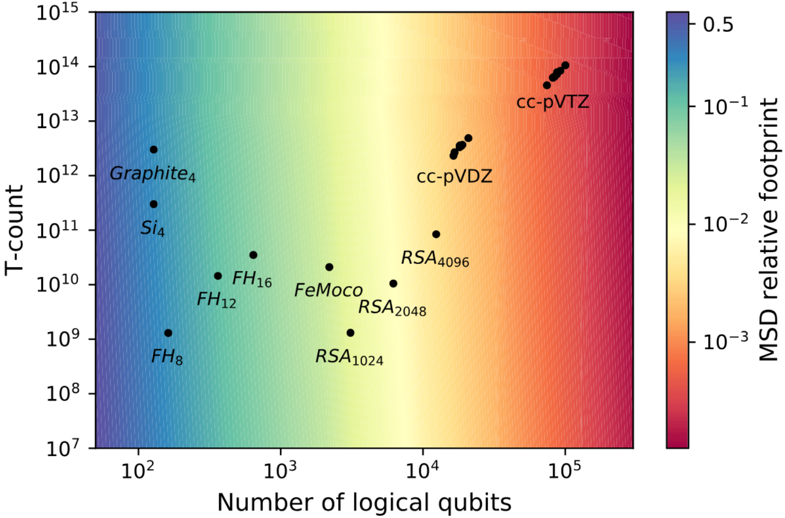


Figure: Ratio of magic state distillation (MSD) footprint to total computational footprint for different number of logical qubits and T-counts for fusion-based QC [11].

Fault-Tolerant Quantum Computing - Resource Estimation

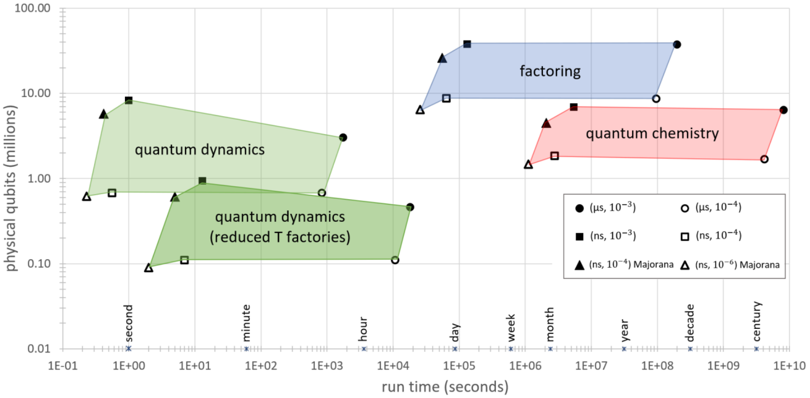


Figure: Runtime of 3 applications for different gate times and modalities: superconducting [ns], ion-traps [μs], and Majorana [3].

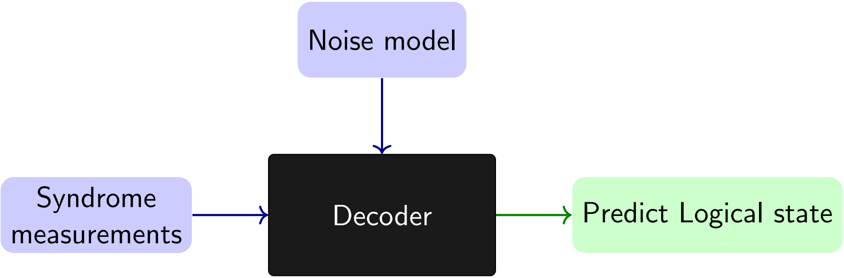
Observation: With 100 [μs] gate times, large algorithms will take almost a year!

Outline

The Decoder

Goal: Determine the state of the logical qubit

Input: Syndrome measurements and noise information
Output: Logical state estimate



The Backlog Problem

Non-Clifford operations (e.g. T-gates) require processing of *all* prior syndrome measurements

When Decoder slower than syndrome generation rate:

1. Define rates:
   - r_{gen}: syndrome generation rate
   - r_{proc}: syndrome processing rate
2. Let $f = \frac{r_{gen}}{r_{proc}} \geq 1$ (backlog factor)
3. For initial T-gate (T_0):
   - Processing overhead time: Δ_{gen}
   - New syndromes during processing: $D_1 = r_{gen} \times \Delta_{gen}$

Observation

Terhal [17] showed: Overhead for k-th T-gate grows as $f^k D_1$

The Backlog Problem

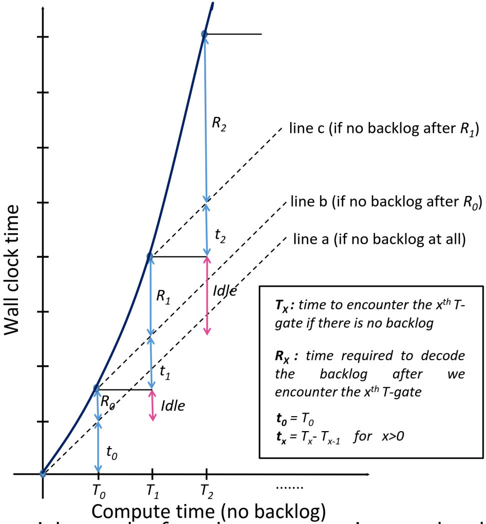


Figure: Exponential growth of syndrome processing overhead for $f > 1$ [10].

The Backlog Problem - Example

Circuit Parameters

- Logical qubits: 100
- Total gates: 2,356
- T-gates: 686

Timing Parameters

- Syndrome generation cycle: 400 [ns]
- Decoder processing time: 800 [ns]
- Backlog factor $f = \frac{r_{gen}}{r_{proc}} = 2$

Circuit execution time: $\mathbf{10^{196}}$ seconds!

Decoder speed and **T-gate count** critical metrics for practical quantum computation.

Real-Time Decoding for Superconducting QPU

Real-time decoding challenging for superconducting devices due to gate speed: Cycle time < 1 [μs].

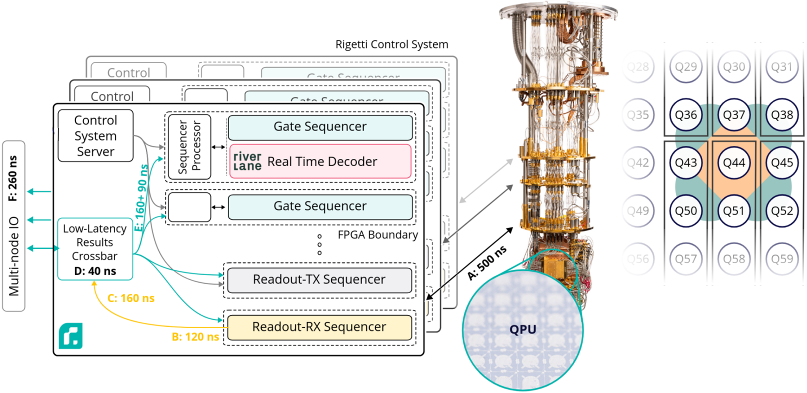


Figure: Integration of Riverlane's FPGA decoder into Rigetti's control system. Latencies are represented by edge labels. Demonstrate mean decoding time below 1 [μs] for Rigetti's Ankaa-2 device [5].

Type of Decoders

Many types of decoders exist, with unique properties:

- Maximum-likelihood decoder
  Optimal, but computationally infeasible
- Matching-based: (e.g. MWPM [9] or BP)
  Optimal for independent errors, widely studied
- Clustering-based (e.g. Union Find [6])
  Fast, near-linear time complexity
- Tensor Networks:
  Handles correlations well, higher computational overhead
- Neural Networks: (e.g. AlphaQubit [2])
  Potential for handling complex noise models

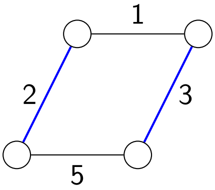
Key trade-off: **decoding speed** vs. **correction accuracy**

We are going to explore *MWPM* and *neural decoders*.

Graph Matching

Perfect Matching Problem: Given a weighted graph $G = (V, E, w)$, where $w : E \to \mathbb{R}$

- Find matching $M \subseteq E$ where each $v \in V$ appears in exactly one edge in M
- Minimize total weight: $\min_M \sum_{e \in M} w(e)$



MWPM Decoder - Idea

Observation: Error chains create distinct syndrome patterns

Types of Error Chains:

1. *Boundary chains*
   - Single syndrome at interior of surface
   - Other end terminates at code boundary
2. *Interior chains*
   - Two syndromes: one at each end

Matching idea:

- Each chain has an associated occurrence probability
- Match all active syndromes minimizing error probability

MWPM Decoder - Example

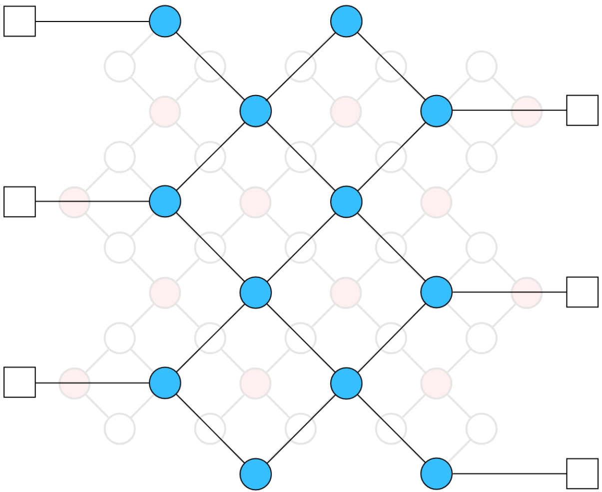


Figure: Tanner graph for Pauli-Z type errors for the distance 5 surface code.

MWPM Decoder - Example

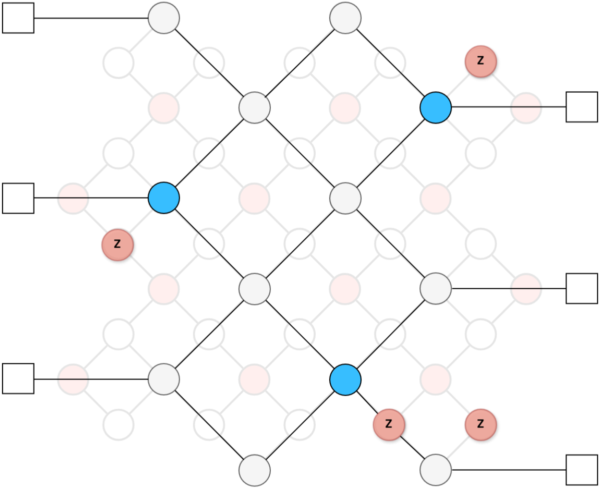


Figure: Active syndromes in Tanner graph for given Pauli-Z errors.

MWPM - Example

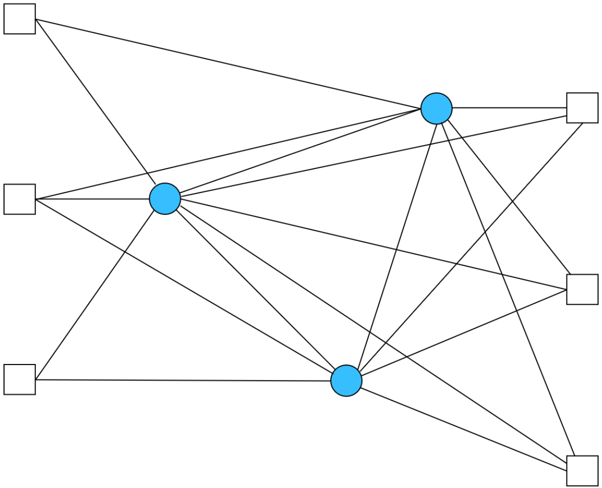


Figure: Syndrome graph for active syndromes.

MWPM - Example

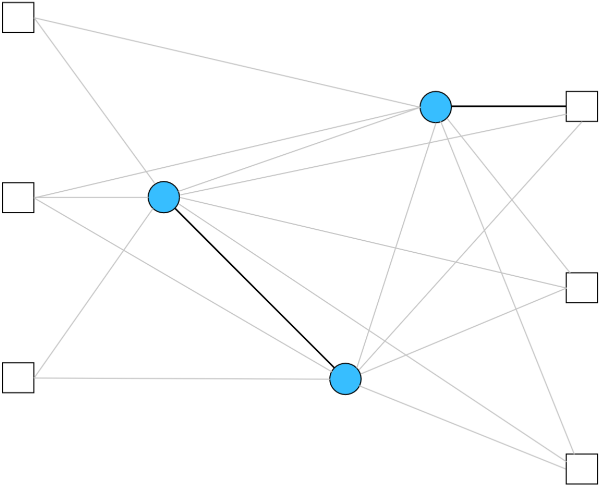


Figure: Matched syndrome graph for active syndromes.

MWPM - Example

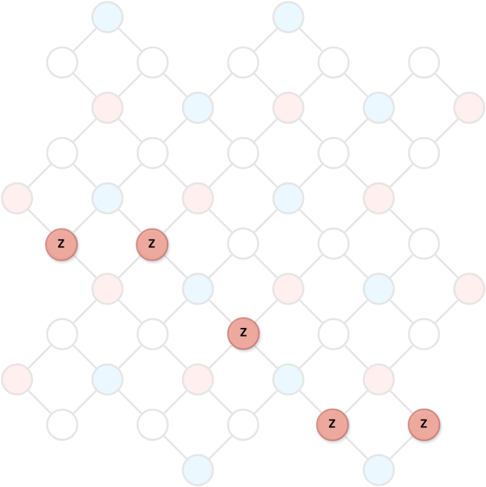


Figure: Decoded errors leading to logical Z_L error by connecting chain of Pauli-Z errors to opposite boundaries.

MWPM Decoder - Construction

Setup:

- Decode Pauli-Z and X errors separately
- Consider **independent** Z errors $E \in \{I, Z\}^n$ for CSS code

For stabilizer generator set $\{S_i\}_i$ define:

- Syndrome bits: $s_i \in \{0, 1\}$, where $s_i = 1$ if generator S_i anti-commutes with E
- Error vector: $e \in \{0, 1\}^n$ if $E_i = Z_i$

Error Probability:

$$p(E) = \prod_i (1 - p_i)^{(1-e_i)} \cdot p_i^{e_i} = \prod_i (1 - p_i) \prod_i \left(\frac{p_i}{1 - p_i}\right)^{e_i}$$

Use logarithmic form, avoiding numerical issues:

$$\log(p(E)) = \sum_i \log(1 - p_i) - \sum_i w_i \cdot e_i,$$

where $w_i = \log((1 - p_i)/p_i)$

MWPM Decoder - Construction

Graph Construction:

- Condition: Each Z-error anti-commutes with two X-stabilizers
- Define matching graph $G = (V, E)$ with $|V| = |s|$
- $(v, w) \in E$, if S_v and S_w anti-commute with Pauli-Z on qubit
- Set edge weight to w_i for qubit i

Decoding Strategy:

- *Perfect matching*: Match all nodes with $s_i = 1$
- *Minimum-weight*: Find smallest chain with $s_i = 1$ at boundaries
  $\rightarrow$ More probable errors have lower weight

Implementation:

- Matching: Edmond's Blossom algorithm
  - Complexity: $\mathcal{O}(|s|^3 \log(|s|))$
- Syndrome graph: Dijkstra's algorithm

Neural Network Decoder - AlphaQubit

QEC's "The Bitter Lesson" moment?

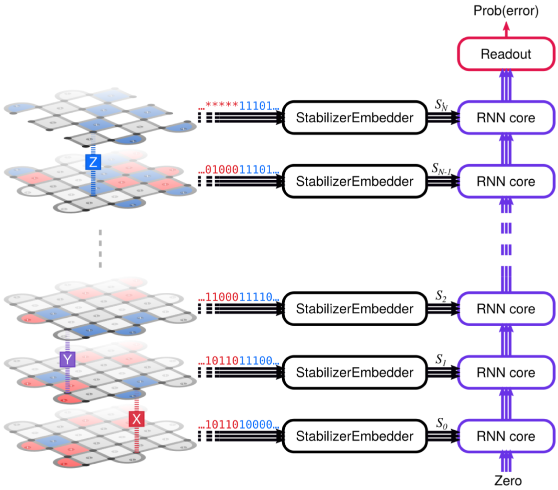


Figure: Decoder's recurrent network structure. Syndromes update transformer state. Outputs single-bit, indicating if logical bit was flipped. Evaluated up to code distance 11 [2].

AlphaQubit - Training

Pretraining

- 2.5 billion samples from 3 sources:
  1. SI1000: 25 QEC rounds, no device-fit
  2. Noise estimate from XEB
  3. Noise estimate for Tanner graph weights p_{ij}

Finetuning

- Pauli+ simulator including leakage, analogue readouts, and cross-talk
- 100 million samples

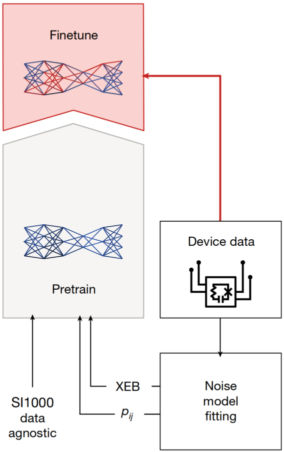


Figure: Training stages [2].

AlphaQubit - Stabilizer Embedding Layer

Input:

- Binary syndrome measurement and temporal differences
- Leakage events and their probability
- Embedded stabilizer index i

Output: $d^2 - 1$ different embeddings

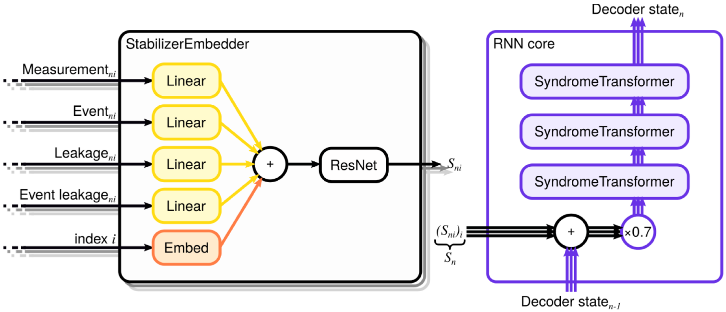


Figure: Stabilizer embeddings used as input for AlphaQubits internal transformer state update round [2].

AlphaQubit - Results

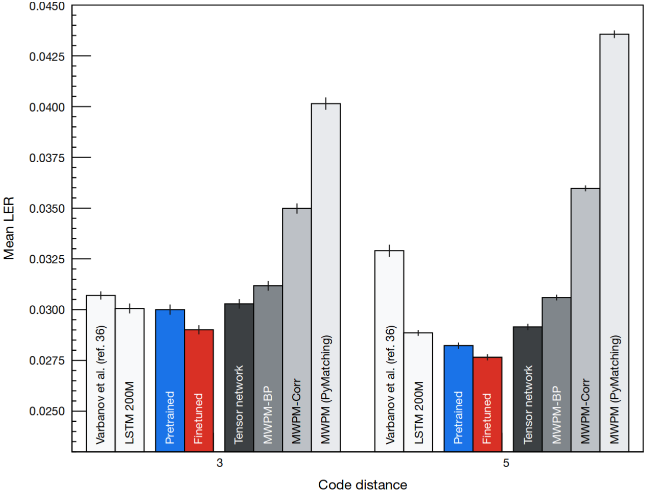


Figure: Mean logical error per QEC round for Surface Code distances 3 and 5 on Google's Sycamore device. Results averaged across bases $\{X, Y, Z\}$ [2].

Decoder Threshold Analysis

Threshold Dependencies:

- Decoder algorithm
- Noise model
- QEC code structure

For distance d, physical error p and threshold p_{thr}:

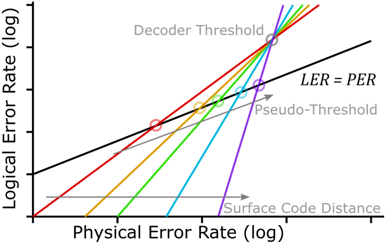


Figure: Threshold example [15].

Logical Error Scaling:

$$\varepsilon_d \propto \left(\frac{p}{p_{thr}}\right)^{\frac{(d+1)}{2}}$$

Error Suppression:

$$\Lambda = \frac{\varepsilon_d}{\varepsilon_{d+2}} \sim \frac{p_{thr}}{p}$$

Note: Threshold comparisons must consider all factors!

Outline

Google - Surface Code Experiment

Quantum Memory Experiment:
Preserve logical qubit for many QEC cycles

Setup:

- Sycamore: 105-qubits transmon device
- Distances: $d = 3$, 5, and 7
- X/Y 25 [ns], CZ 42 [ns]
- TLS mitigation strategy

Main Results:

- Demonstrate $\Lambda > 2$
- Life-time of logical qubit $2\times$ of best physical qubit on QPU
- Real-time decoding $< 1.1[\mu s]$

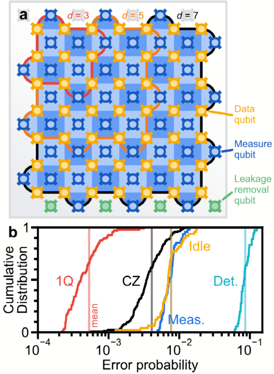


Figure: **a** Sycamore topology. **b** Gate error distribution [1].

Google - Real-Time Decoder Data Flow

1. Control electronics classify I/Q readout into 0/1
2. Transmitted to workstation via low-latency Ethernet
3. Measurements converted to detection events
4. Streamed to constant sized shared-buffer
5. Decoder reads from buffer

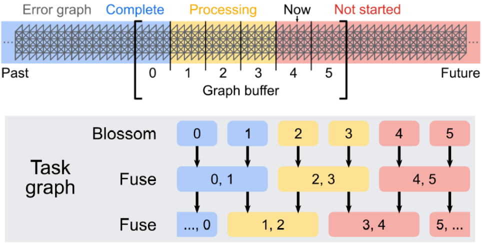


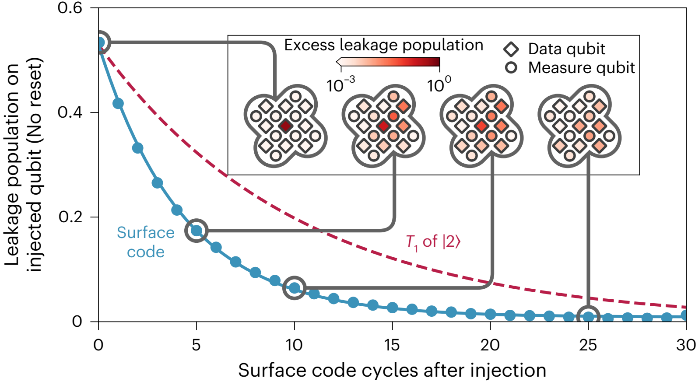
Figure: Windowed streaming decoder: Local Blossom algorithm with subsequent fusing until global MWPM is found [1].

Google - Correlated Errors through Leakage

Transmons not ideal qubits → Leakage to $|2\rangle, |3\rangle, \ldots$ possible.

Problem: QEC assumes uncorrelated errors. Leakage causes correlated errors! Especially **CZ gate** prone to leakage.

DQLR: Use *Leakage iSWAP* to transfer leakage to ancillas [14].



Google - Results

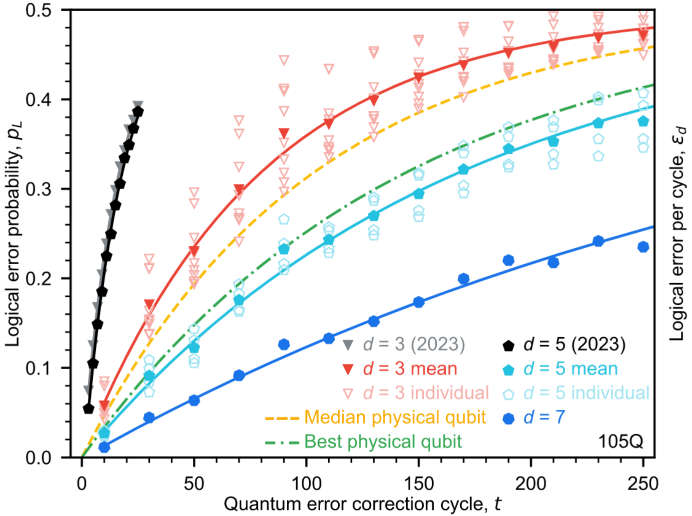


Figure: Logical error rates (LER) over multiple QEC cycles demonstrating $\Lambda = 2.14 \pm 0.02$ [1].

Quantum Error Correction - Summary

Many more topics ...

- Quantum LDPC codes, color codes, ...
- Subsystem codes
- Bosonic codes
- Quantum resource estimation
- ...

An Interdisciplinary Field!

- QPU fabrication and control
- Software development and tooling
- Novel error correction code design

Theory Meets Practice

- Transition from theory to implementation
- Emerging real-world demonstrations

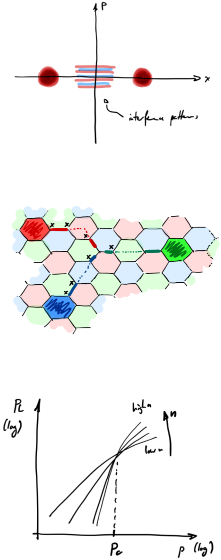


Figure: Source: [8]

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